(Time: 2½ hours)

Total Marks: 75

N. B.: (1) **All** questions are **compulsory**.
(2) Make **suitable assumptions** wherever necessary and **state the assumptions** made.
(3) Answers to the **same question** must be **written together**.
(4) Numbers to the **right** indicate **marks**.
(5) Draw **neat labeled diagrams** wherever **necessary**.
(6) Use of **Non-programmable** calculators is **allowed**.

1.	**Attempt *any three* of the following:**	**15**
a.	Reduce the matrix to normal form and find its rank where $A=\begin{bmatrix}1 & -1 & 3 & 6\\ 1 & 3 & -3 & -4\\ 5 & 3 & 3 & 11\end{bmatrix}$ Given matrix is, $$A=\begin{bmatrix}1 & -1 & 3 & 6\\ 1 & 3 & -3 & -4\\ 5 & 3 & 3 & 11\end{bmatrix}$$ Operate $R_2-R_1;\ R_3-5R_1 \sim \begin{bmatrix}1 & -1 & 3 & 6\\ 0 & 4 & -6 & -10\\ 0 & 8 & -12 & -19\end{bmatrix}$ Operate $C_2+C_1;\ C_3-3C_1;$ $C_4-6C_1;\ \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 4 & -6 & -10\\ 0 & 8 & -12 & -19\end{bmatrix}$ Operate $\frac{C_2}{4};\frac{C_3}{-6} \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & -10\\ 0 & 2 & 2 & -19\end{bmatrix}$ Operate $R_3-2R_2 \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & -10\\ 0 & 0 & 0 & 1\end{bmatrix}$ Operate $C_3-C_2;$ $C_4+10C_2 \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ $\sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ Operate $C_{343} \sim [\ I_3 \quad 0\ ]$ This a normal form of matrix A. **∴ Rank of matrix = $\rho(A) = 3$**	
b.	Examine for consistency the system of equations $x-y-z=2;\quad x+2y+z=2;\quad 4x-7y-5z=2$ and solve them if found consistence.	

AX = B

Since here 3 Equations and 3 unknowns : so first check | A |

$$\therefore |A| = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix}$$

$$= 1[-3] + 1[-9] - 1[-15] = 3$$

$\therefore |A| \neq 0$ i.e. A^{-1} exist.

$$\therefore X = A^{-1}B \quad \ldots(1)$$

Now,

$$A^{-1} = \frac{1}{|A|} \text{ adj. A.}$$

Co-factor of each element of

$$A = \begin{bmatrix} -3 & 9 & -15 \\ 2 & -1 & 3 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\therefore \text{Adj. A} = \begin{bmatrix} -3 & 2 & 1 \\ 9 & -1 & -2 \\ -15 & 3 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & 2 & 1 \\ 9 & -1 & -2 \\ -15 & 3 & 3 \end{bmatrix}$$

Put this in Equation (1)

$$\therefore X = \frac{1}{3}\begin{bmatrix} -3 & 2 & 1 \\ 9 & -1 & -2 \\ -15 & 3 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$X = \frac{1}{3}\begin{bmatrix} 0 \\ 12 \\ -18 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -6 \end{bmatrix}$$

$\therefore x = 0, y = 4, z = -6$ is a solution.

c. Verify cayley – Hamilton Theorem for the matrix A.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

The characteristic equation is

$$\begin{bmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{bmatrix} = 0$$

after simplification, we get

$$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

Cayley - Hamilton theorem states that this equation is satisfied by the matrix A.

i.e $A^3 - 5A^2 + 9A - I = 0 \quad \ldots(1)$

Now

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$\text{and} \quad A^3 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

It can be easily seen that

$$A^3 - 5A^2 + 9A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

d. Express in Polar from $-1+\sqrt{3}\,i$

By comparing the complex number with standard form.

$z = x + iy$, we get $x = -1$ and $y = \sqrt{3}$ then

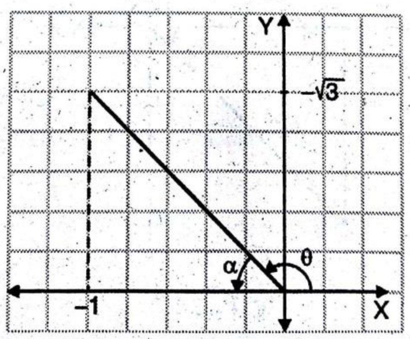
Modulus $= |z| = \sqrt{x^2+y^2} = \sqrt{(-1)^2 + (\sqrt{3}^2)} = 2$

Amplitude $= \theta = \pi - \alpha$

Where, $\alpha = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$= \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

Amplitude $= \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$



e. Simplify $\dfrac{(\cos\theta - \sin\theta)^6 (\cos 5\theta - i\sin 5\theta)^{-2}}{(\cos 8\theta + i\sin 8\theta)^{1/2}}$ using De-Moivre's theorem.

	Soln. : Consider, $$z = \frac{(\cos\theta - i\sin\theta)^6(\cos 5\theta - i\sin 5\theta)^{-2}}{(\cos 8\theta + i\sin 8\theta)^{1/2}} \quad \ldots(1)$$ We know, De-Moivre's theorem. $$\left.\begin{aligned}(\cos\theta + i\sin\theta)^n &= \cos n\theta + i\sin n\theta\\ (\cos\theta + i\sin\theta)^{-n} &= \cos n\theta - i\sin n\theta\end{aligned}\right\} \quad \ldots(2)$$ Using this write each term of Equation (1) with base $(\cos\theta + i\sin\theta)$ $$\therefore\ z = \frac{(\cos\theta + i\sin\theta)^{-1\times 6}(\cos\theta + i\sin\theta)^{-5\times(-2)}}{(\cos\theta + i\sin\theta)^{8\times\frac{1}{2}}}$$ $$z = \frac{(\cos\theta + i\sin\theta)^{-6}(\cos\theta + i\sin\theta)^{10}}{(\cos\theta + i\sin\theta)^4}$$ $$\therefore\ z = (\cos\theta + i\sin\theta)^{-6+10-4} \text{(By Laws of indices)}$$ $$z = (\cos\theta + i\sin\theta)^0 = \cos 0 + i\sin 0$$ [By Equation (2)] $$\mathbf{z = 1 \text{ or } z = 1 + 0i} \quad (\because \sin 0 = 0 \text{ and } \cos 0 = 1)$$ $$\therefore\ \frac{(\cos\theta - i\sin\theta)^6(\cos 5\theta - i\sin 5\theta)^{-2}}{(\cos 8\theta + i\sin 8\theta)^{1/2}} = 1$$	
f.	Prove that : $\therefore \sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$ Considery $= \sinh^{-1} x$ $$x = \sinh y = \frac{e^y - e^{-y}}{2}$$ $$\therefore 2x = e^y - e^{-y} \Rightarrow$$ $$2x \cdot e^y = e^{2y} - 1 \quad \therefore e^{2y} - 2x\,e^y - 1 = 0$$ Which is a quadratic equation in e^y, $$\therefore\ e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$ $$= \frac{2x \pm 2\sqrt{x^2+1}}{2}$$ $$= x \pm \sqrt{x^2+1}$$ Taking only positive value, $$e^y = x + \sqrt{x^2+1}$$ $$\therefore\ y = \log(x + \sqrt{x^2+1})$$ $$\therefore\ \mathbf{\sinh^{-1} x = \log(x + \sqrt{x^2+1})}$$	
2.	**Attempt *any three* of the following:**	**15**
a.	Solve $y^2 - x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}$	

	Consider the differential equation $$y^2 - x^2\frac{dy}{dx} = xy\frac{dy}{dx}$$ $$\therefore\ xy\frac{dy}{dx} + x^2\frac{dy}{dx} = y^2$$ $$\frac{dy}{dx}(xy + x^2) = y^2$$ $$\therefore \frac{dy}{dx} = \frac{y^2}{xy + x^2}$$ is the homogeneous differential equation Put $y = Vx$ $$\therefore \frac{dy}{dx} = V + x\frac{dV}{dx}$$ $\therefore$ The differential equation becomes, $$V + x\frac{dV}{dx} = \frac{(Vx)^2}{x(Vx) + x^2}$$ $$\therefore\ x\frac{dV}{dx} = \frac{V^2x^2}{x^2V + x^2} - V$$ $$x\frac{dV}{dx} = \frac{V^2}{V+1} - V$$ $$x\frac{dV}{dx} = \frac{V^2 - V^2 - V}{V+1}$$ $$x\frac{dV}{dx} = \frac{-V}{1+V}$$ $$\therefore \frac{1+V}{V}dV = -\frac{dx}{x}$$ is the variable separable form On integrating both sides, $$\int\frac{1+V}{V}dV = -\int\frac{dx}{x}$$ $$\int\left(\frac{1}{V}+1\right)dV = -\int\frac{dx}{x}$$ $$\therefore\ \log\lvert V\rvert + V = -\log\lvert x\rvert + C$$ $$\log\lvert V\rvert + \log\lvert x\rvert + V = C$$ $$\log[\lvert V\rvert . \lvert x\rvert] + V = C$$ $$\log\left(\frac{y}{x}\times x\right) + \frac{y}{x} = C$$ $$\therefore\ \log\lvert y\rvert + \frac{y}{x} = C$$ is the general solution.	
b.	Solve $\frac{dy}{dx} + 2y\tan x = \sin x$	

Consider, the given differential equation

$$\frac{dy}{dx} + 2y \tan x = \sin x$$

Comparing with $\frac{dy}{dx} + Py = Q$

$P = 2\tan x$ and $Q = \sin x$

$$\text{I.F.} = e^{\int P\,dx} = e^{2\int \tan x\,dx}$$

$$= e^{2\int \frac{\sin x}{\cos x}dx}$$

$$\text{I.F.} = e^{2\log \sec x} = e^{\log \sec^2 x}$$

$$\text{I.F.} = \sec^2 x$$

Then the general solution is,

$$y\,(\text{I.F.}) = \int Q\cdot(\text{IF})\,dx + C = \int \sin x \cdot \sec^2 x\,dx + C$$

$$= \int \sin x \cdot \frac{1}{\cos^2 x}\,dx + C = \frac{\sin x}{\cos x}\frac{1}{\cos x}dx + C$$

$$= \int \sec x \cdot \tan x\,dx + C$$

$$y \sec^2 x = \sec x + C$$

is required general solution.

c. Solve $(p-2x)(p-y)=0$

Consider $(p-2x)(p-y) = 0$

$$p - 2x = 0 \text{ or } p - y = 0$$

(i) Consider

$$p - 2x = 0$$

$$\frac{dy}{dx} = 2x$$

$$dy = 2x\,dx$$

On integrating, $\int dy = 2\int x\,dx$

$$y = x^2 + C$$

$$y - x^2 - C = 0$$

(ii) $p - y = 0$

$$\therefore \frac{dy}{dx} = y$$

$$\frac{dy}{y} = dx$$

On integrating, $\frac{1}{y}dy = \int dx$

$$\log y = x + C$$

$$\log y - x - C = 0$$

$\therefore$ The general solution is

$(y - x^2 - C)(\log y - x - C) = 0$

d. Solve : $y = xp + \frac{1}{p}$

	$y = xp + \frac{1}{p}$ which is already clairaut's equation with $f(p) = \frac{1}{p}$ and $f'(p) = -\frac{1}{p^2}$ Hence the solution is $y = CX + \frac{1}{C}, C \neq 0$ is an arbitrary constant. $x + f'(p) = 0$ $x - \frac{1}{p^2} = 0$ Eliminating P between the equation $x - \frac{1}{p^2} = 0$ and the equation $y = xp + \frac{1}{p}$ yields $y = x\frac{1}{\sqrt{x}} + \sqrt{x} = \frac{x+x}{\sqrt{x}} = \frac{2x}{\sqrt{x}}$ $y = 2\sqrt{x}$ $y^2 = 4x$ Which is the required singular solution of the equation $y = xp + \frac{1}{p}$	
e.	Solve : $(D^2 + 6D + 9)y = 5^x - \log 2$ The complete solution of this differential equation is, $y = y_c + y_p = C.F. + P.I.$ **Step I : C. F. :** A. E.(Auxiliary Equation) is, $D^2 + 6D + 9 = 0$ $(D+3)^2 = 0$ $D = -3, -3$ $y_c = (C_1 + C_2 x)e^{-3x}$ **Step II : P.I.** $y_p = \frac{1}{D^2 + 6D + 9}(5^x - \log 2)$ $= \frac{1}{(D+3)^2} 5^x - \frac{1}{(D+3)^2} (\log 2) e^{0x}$ Replace : (D = log 5) (D = 0) $y_p = \frac{5^x}{(\log 5 + 3)^2} - \frac{\log 2}{9}$ **Step III :** Hence the complete solution is, $y = y_c + y_p$ $y = (C_1 + C_2 x)e^{-3x} + \frac{5^x}{(\log 5 + 3)^2} - \frac{\log 2}{9}$	
f.	Solve : $x^2 \frac{d^2 y}{dx^2} - x\frac{dy}{dx} - 3y = 0$	

	Step I : Given differential equation is Cauchy's linear differential equation, Put $x = e^z \Rightarrow z = \log x$ and $D \equiv \frac{d}{dz}$ $\therefore$ Differential Equation becomes, $[D(D-1) - D - 3]\,y = 0$ $(D^2 - D - D - 3)\,y = 0$ $(D^2 - 2D - 3)\,y = 0$ The complete solution of this differential equation is, $y = y_c + y_p = \text{C.F.} + \text{P.I.}$ **Step II : C.F. :** A.E. (Auxiliary Equation) is, $D^2 - 2D - 3 = 0$ $(D-3)(D+1) = 0$ $D = 3, -1$ $y_c = C_1 e^{3z} + C_2 e^{-z}$ ...(1) **Step III : P.I. :** $y_p = 0$ ...(2) **Step IV :** Hence, $y = y_c + y_p$ $y = C_1 e^{3z} + C_2 e^{-z}$ ...[From Equations (1) and (2)] $\mathbf{y = C_1 x^3 + C_2 x^{-1}}$ $(\because e^z = x)$ This is required solution.	
3.	**Attempt *any three* of the following:**	**15**
a.	Find the Laplace transform of $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$ $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$ Given, $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$ $\therefore L[f(t)] = \int_0^\pi e^{-st}(\cos t)\,dt + \int_\pi^\infty e^{-st}(\sin t)\,dt$ $= \left[\frac{e^{-st}}{s^2+1} [-s\cos t + \sin t] \right]_0^\pi + \left[\frac{e^{-st}}{s^2+1} [-s\sin t - \cos t] \right]_\pi^\infty$ $= \left[\left[\frac{e^{-\pi s}}{s^2+1}(s) \right] - \left[\frac{1}{s^2+1}(-s) \right] \right] + \left[\left[[0] - \frac{e^{-\pi s}}{s^2+1}(1) \right] \right]$ $= \frac{se^{-\pi s}}{s^2+1} + \frac{s}{s^2+1} - \frac{e^{-\pi s}}{s^2+1}$ $= \frac{e^{-\pi s}}{s^2+1}(s-1) + \frac{s}{s^2+1}$ $\therefore L[f(t)] = \frac{1}{s^2+1}[e^{-\pi s}(s-1) + s]$ $= \frac{[e^{-\pi s}(s-1) + s]}{s^2+1}$	
b.	Evaluate by using Laplace transform $\int_0^\infty t^2 e^{-t} \sin t\, dt$	

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \int_0^\infty t^2 e^{-t} \sin t\,dt = \int_0^\infty e^{-t}[t^2 \sin t]\,dt$$

$$= L[t^2 \sin t] = (-1)^2 \frac{d^2}{ds^2}[L(\sin t)] \quad (s = 1)$$

$$= \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right] = \frac{d}{ds}\left[\frac{-1}{(s^2+1)^2}(2s)\right]$$

$$= \frac{d}{ds}\left[\frac{-2s}{(s^2+1)^2}\right] \quad (s = 1)$$

$$\therefore \int_0^\infty t^2 e^{-t} \sin t\,dt = \left[\frac{(s^2+1)^2(-2) - (-2s)\,2(s^2+1)(2s)}{(s^2+1)^4}\right] \quad (s = 1)$$

$$= \frac{2(s^2+1)[-(s^2+1)+4s^2]}{(s^2+1)^4}$$

$$= \frac{2(3s^2-1)}{(s^2+1)^3} \quad (s = 1)$$

$$\therefore \int_0^\infty t^2 e^{-t} \sin t\,dt = \frac{2(3-1)}{(1+1)^3} = \frac{1}{2} \quad (s = 1)$$

$$\int_0^\infty t^2 e^{-t} \sin t\,dt = \frac{1}{2}$$

c. Find the Laplace transform of the following.

$$\frac{dy}{dt} + 3y(t) + 2\int_0^t y(t)\,dt = t; \quad given\ y(0) = 0$$

Given differential equation,

$$\frac{dy}{dt} + 3y(t) + 2\int_0^t y(t)\,dt = t; \quad y(0) = 0$$

Taking Laplace Transform of both side

$$L\left[\frac{dy}{dt} + 3y(t) + 2\int_0^t y(t)\,dt\right] = L[t]$$

$$s\,y(s) - y(0) + 3\,y(s) + 2\frac{1}{s}y(s) = \frac{1}{s^2};$$

$$\left(s + 3 + \frac{2}{s}\right) y(s) - y(0) = \frac{1}{s^2}$$

$$\left(\frac{s^2+3s+2}{s}\right) y(s) - 0 = \frac{1}{s^2};$$

$$\therefore L[y(t)] = y(s) = \frac{1}{s(s^2+3s+2)}$$

d. Find the inverse Laplace transform of $\frac{s}{(s-2)^4}$

$$\frac{s}{(s-2)^4} = \frac{s-2+2}{(s-2)^4}$$

$$= \frac{1}{(s-2)^3} + \frac{2}{(s-2)^4}$$

$$\therefore L^{-1}\left[\frac{s}{(s-2)^4}\right] = e^{2t} L^{-1}\left[\frac{1}{s^3} + \frac{2}{s^4}\right]$$

$$= e^{2t}\left[\frac{t^2}{2!} + 2\frac{t^3}{3!}\right] = e^{2t}\left[\frac{t^2}{2} + \frac{t^3}{3}\right]$$

e. Find inverse Laplace transform of $\cot^{-1}(s)$

Consider, $L^{-1}[\cot^{-1}(s)] = f(t)$

$$\therefore \quad L[f(t)] = \cot^{-1}(s)$$

$$\therefore \quad L[t\,f(t)] = \frac{-d}{ds}[\cot^{-1}(s)]$$

$$= -\left[\frac{-1}{s^2+1}\right]$$

$$\therefore \quad L[t\,f(t)] = \frac{1}{s^2+1}$$

$$\therefore \quad t\,f(t) = L^{-1}\left[\frac{1}{s^2+1}\right]$$

$t\,f(t) = \sin t$ (∵ by Table 7.2.1)

$$\therefore \quad f(t) = \frac{\sin t}{t}$$

$$\therefore \quad \mathbf{L^{-1}[\cot^{-1}(s)] = \frac{\sin t}{t}}$$

f. Find the Laplace transform of : $f(t) = \begin{cases} 1 & 0 < t < a \\ -1 & a < t < 2a \end{cases}$ and $f(t) = f(t+2a)$

Soln. : Given, $f(t) = \begin{cases} 1 & 0 < t < a \\ -1 & a < t < 2a \end{cases}$

and $f(t) = f(t+2a)$

This is periodic function with period T = 2a

We have, The Laplace transform of periodic function,

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-su} f(u)\,du$$

$$\therefore L[f(t)] = \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-su}(1)\,du + \int_a^{2a} e^{-su}(-1)\,du\right]$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[\frac{e^{-su}}{-s}\right]_0^a - \left[\frac{e^{-su}}{-s}\right]_a^{2a}\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[\frac{e^{-as}}{-s} - \frac{1}{-s}\right] - \left[\frac{e^{-2as}}{-s} - \frac{e^{-as}}{-s}\right]\right\}$$

$$= \frac{1}{(-s)(1-e^{-2as})}\left[e^{-as} - 1 - e^{-2as} + e^{-as}\right].$$

$$= \frac{-e^{-2as} + 2e^{-as} - 1}{(-s)(1-e^{-2as})} = \frac{1 - 2e^{-as} + e^{-2as}}{s(1-e^{-2as})}$$

$$= \frac{(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} = \frac{(1-e^{-as})}{s(1+e^{-as})}$$

$$\therefore \quad L[f(t)] = \frac{(1-e^{-as})}{s(1+e^{-as})}$$

4.	**Attempt *any three* of the following:**	**15**
a.	Evaluate : $\int_0^1\int_0^y xy\, e^{-x^2} dx\, dy$ **Step I : Consider,** $I = \int_0^1\int_0^y xy\, e^{-x^2} dx\, dy$ Limits of inner integral are functions of x ∴ Integrate w.r.t. x first. $I = \int_0^1 \left[\int_0^y xy\, e^{-x^2} dx\right] dy$ **Step II :** $I = \int_0^1 y\left(\frac{-1}{2}\right)\left[\int_0^y e^{-x^2}(-2x)\, dx\right] dy$ ...[Note this adjustment] $I = \frac{-1}{2}\int_0^1 y\left[e^{-x^2}\right]_0^y dy \quad \left[\because \int e^{f(x)} f'(x)\, dx = e^{f(x)}\right]$ $I = \frac{-1}{2}\int_0^1 y\left[e^{-y^2} - 1\right] dy = \frac{-1}{2}\int_0^1 \left[e^{-y^2} y - y\right] dy$ $= \frac{-1}{2}\left\{\frac{-1}{2}\int_0^1 e^{-y^2}(-2y)\, dy - \left[\frac{y^2}{2}\right]_0^1\right\}$ $= \frac{-1}{2}\left\{\frac{-1}{2}\left[e^{-y^2}\right]_0^1 - \left[\frac{1}{2}\right]\right\}$ $= \frac{1}{4}\left[e^{-1} - 1\right] + \frac{1}{4} = \frac{1}{4}\left[e^{-1} - 1 + 1\right] = \frac{1}{4e}$ ∴ $I = \frac{1}{4e}$	
b.	Take Expression as a single integral and evaluate $\int_0^{a/\sqrt{2}}\int_0^x x\, dx\, dy + \int_{a/\sqrt{2}}^a \int_0^{\sqrt{a^2-x^2}} x\, dx\, dy$ **Step I : Consider,** $I = \int_0^{\frac{a}{\sqrt{2}}}\int_0^x x\, dx\, dy + \int_{\frac{a}{\sqrt{2}}}^a \int_0^{\sqrt{a^2-x^2}} x\, dx\, dy$ ...(1) $I = I_1 + I_2$ The limits of I_1 are y = 0, y = x and x = 0 and $x = a\sqrt{2}$ and limits of I_2 are y = 0, $y = \sqrt{a^2 - x^2}$ and $x = a\sqrt{2}$; x = a. The region of integration of I_1 and I_2 is as shown in Fig. P. 8.3.9 since point of integration of $x^2 + y^2 = a^2$ and y = x is $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$ To consider the total region (I_1 and I_2) take a horizontal strip SR Along a strip x varies from x = y to $x = \sqrt{a^2 - y^2}$ and y varies from y = 0 to $y = \frac{a}{\sqrt{2}}$	

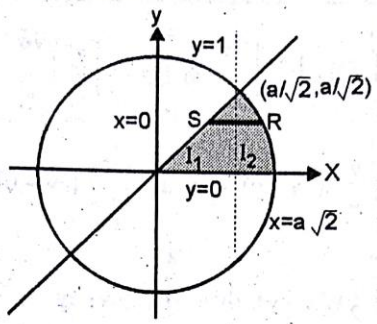


Fig. P. 8.3.9

Step II :

$$\therefore \quad I = \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} x\,dx\,dy = \int_0^{a/\sqrt{2}} dy \int_y^{\sqrt{a^2-y^2}} x\,dx$$

$$= \int_0^{a/\sqrt{2}} dy \left[\frac{x^2}{2}\right]_y^{\sqrt{a^2-y^2}} = \frac{1}{2}\int_0^{a/\sqrt{2}} (a^2 - y^2 - y^2)\,dy$$

$$= \frac{1}{2}\int_0^{a/\sqrt{2}} (a^2 - 2y^2)\,dy$$

$$= \frac{1}{2}\left[a^2 y - 2\frac{y^3}{3}\right]_0^{\frac{a}{\sqrt{2}}} = \frac{1}{2}\left[\frac{a^3}{\sqrt{2}} - \frac{2}{3}\left(\frac{a}{\sqrt{2}}\right)^3\right]$$

$$= \frac{1}{2} a^3\left[\frac{1}{\sqrt{2}} - \frac{2}{3}\,\frac{1}{2\sqrt{2}}\right] = \frac{a^3}{2}\frac{1}{\sqrt{2}}\left(1 - \frac{1}{3}\right) = \frac{a^3}{3\sqrt{2}}$$

$$\therefore \mathbf{I} = \frac{a^3}{3\sqrt{2}}$$

c. Evaluate $\int_0^{a}\int_0^{\sqrt{a^2-y^2}} \left(\sqrt{a^2 - x^2 - y^2}\right) dx\,dy$

Step I : Consider, $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} \left(\sqrt{a^2-x^2-y^2}\right) dx\,dy$

Convert this integral into polar co-ordinate by using polar transformation, $x = r\cos\theta$; $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$.

Here limits of integration are,

$x = 0$ to $x = \sqrt{a^2-y^2}$ and $y = 0$ to $y = a$

$x = \sqrt{a^2-y^2}$ gives $x^2 = a^2 - y^2 \Rightarrow x^2 + y^2 = a^2$

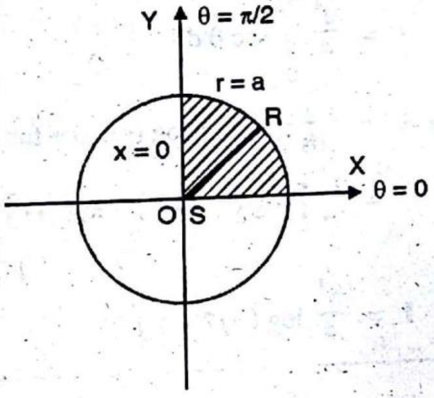


Fig. P. 8.4.7

$\therefore$ In polar :

$x^2 + y^2 = a^2 \Rightarrow r^2 = a^2 \Rightarrow r = a$

$\therefore$ Region of integration is as shown in Fig. P. 8.4.7

Take a radial strip SR, along the strip θ constant and r varies from $r = 0$ to $r = a$

Now, turning the strip throughout the region

$\therefore$ θ varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2}\int_0^{a} \left(\sqrt{a^2-r^2}\right) r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^{a} \left(\sqrt{a^2-r^2}\right) r\,dr\right] d\theta$$

	Put $a^2 - r^2 = t$ $\therefore r^2 = a^2 - t$ $\therefore 2\,r\,dr = -dt$ | r | 0 | a | | t | a^2 | 0 | **Step II :** $I = \int_0^{\pi/2}\left[\int_{a^2}^{0}\sqrt{t}\left(-\frac{dt}{2}\right)\right]d\theta$ $= \frac{1}{2}\int_0^{\pi/2}\left[\int_0^{a^2}\sqrt{t}\,dt\right]d\theta \quad \left[\therefore \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$ $= \frac{1}{2}\int_0^{\pi/2}\left[\frac{t^{3/2}}{3/2}\right]_0^{a^2} d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{2}{3}\left[(a^2)^{3/2}\right]d\theta$ $I = \frac{1}{2}\frac{2}{3}a^3\int_0^{\pi/2} d\theta = \frac{a^3}{3}[\theta]_0^{\pi/2} = \frac{a^3}{3}\left[\frac{\pi}{2}\right]$ $I = \frac{\pi a^3}{6}$	
d.	Evaluate : $\iiint_V \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ where V is the volume bounded by the planes, $x = 0.\ y = 0,\ z = 0, and\ x + y + z = 1.$ Let $I = \iiint_V \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ $= \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} (1+x+y+z)^{-3}\cdot dz$ $= \int_0^1 dx \int_0^{1-x} dy\left[\frac{(1+x+y+z)^{-2}}{-2}\right]_0^{1-x-z}$ $= -\frac{1}{2}\int_0^1 dx \int_0^{1-x} dy\,[(2)^{-2} - (1+x+y)^{-2}]$ $= -\frac{1}{2}\int_0^1 dx\left[\frac{1}{4}\int_0^{1-x} dy - \int_0^{1-x}(1+x+y)^{-2}\cdot dy\right]$ $= -\frac{1}{2}\int_0^1 dx\left[\frac{1}{4}(-x) - \left[\frac{(1+x+y)^{-1}}{-1}\right]_0^{1-x}\right]$ $= -\frac{1}{2}\int_0^1\left[\frac{1}{4}(1-x) + \frac{1}{2} - \frac{1}{1+x}\right]dx$ $= -\frac{1}{2}\left[\frac{1}{4}\left(x - \frac{x^2}{2}\right) + \frac{x}{2} - \log(1+x)\right]_0^1$ $= -\frac{1}{2}\left[\frac{1}{4}\left(1 - \frac{1}{2}\right) + \frac{1}{2} - \log 2\right]$ $= -\frac{1}{2}\left[\frac{5}{8} - \log 2\right] = \frac{1}{2}\log 2 - \frac{5}{16}$	
e.	Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between curve $y = x^2$ and the line $y = x$	

Step I : Consider

$$I = \iint xy\,(x+y)\,dx\,dy$$

Here region of integration is the area between curve $y = x^2$ and the line $y = x$

$\therefore$ Region of integration is as shown in Fig. P. 9.2.3

Take vertical strip SR as shown in Fig. P. 9.2.3.

$\therefore$ Limits are, $y = x^2$ to $y = x$

$x = 0$ to $x = 1$

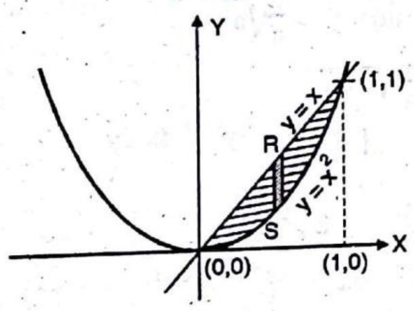


Fig. P. 9.2.3

Step II : $\therefore \; I = \int_0^1 \left[\int_{x^2}^{x} xy\,(x+y)\,dy \right] dx$

$$I = \int_0^1 x \left[\int_{x^2}^{x} (xy + y^2)\,dy \right] dx$$

$$= \int_0^1 x \left[x\frac{y^2}{2} + \frac{y^3}{3} \right]_{x^2}^{x} dx$$

$$= \int_0^1 x \left[\frac{x^3}{2} + \frac{x^3}{3} - \frac{x^5}{2} - \frac{x^6}{3} \right] dx$$

$$I = \int_0^1 x \left[\frac{5x^3}{6} - \frac{x^5}{2} - \frac{x^6}{3} \right] dx$$

$$= \int_0^1 \left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right] dx$$

$$= \left[\frac{5}{6}\frac{x^5}{5} - \frac{1}{2}\frac{x^7}{7} - \frac{1}{3}\frac{x^8}{8} \right]_0^1$$

$$= \left\{ \left[\frac{1}{6} - \frac{1}{14} - \frac{1}{24} \right] - 0 \right\}$$

$$\therefore \; I = \frac{3}{56}$$

f. Prove that the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{e^2} = 1$ is $\frac{4\pi}{3}abc$

	Solution: Required volume: $\iiint dxdydz$ over the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$ Put $\frac{x}{a}=X, \frac{y}{b}=Y$ and $\frac{z}{c}=Z$ $\therefore$ $dx = adX, dy = bdY, dz = cdZ$ $\therefore$ Volume = $\iiint abc\, dxdydz$ over the sphere $x^2+y^2+z^2=1$ Changing to spherical coordinates by the relations. $X = r\sin\theta\cos\phi, Y = r\sin\theta.\sin\phi, Z = r\cos\theta$ $\therefore$ Volume = $abc\iiint r^2\sin\theta\, dr\, d\theta\, d\phi$ over the region. $\{(r,\theta,\phi): 0\le r\le 1, 0\le\theta\le\pi, 0\le\phi\le 2\pi\}$ $= abc\int_0^1 r^2 dr\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi} d\phi$ $= abc\left[\frac{r^3}{3}\right]_0^1[-\cos\theta]_0^{\pi}[\phi]_0^{2\pi}$ $= abc\left(\frac{1}{3}\right)(2)(2\pi)$ $= \frac{4}{3}\pi abc.$	
5.	**Attempt *<u>any three</u>* of the following:**	**15**
a.	Evaluate $\int_0^{\infty} x^2 .\, e^{-h^2x^2} .\, dx$ **Soln. :** Consider, $I = \int_0^{\infty} x^2\cdot e^{-h^2x^2}\cdot dx$ **Put,** $h^2\cdot x^2 = t \Rightarrow x^2 = \frac{t}{h^2}$ and $x = \frac{t^{1/2}}{h}$ $dx = \frac{1}{h}\cdot\frac{1}{2}t^{-1/2}\cdot dt$ x: 0 → ∞; t: 0 → ∞ $I = \int_0^{\infty}\left(\frac{t}{h^2}\right)\cdot e^{-t}\cdot\frac{1}{2h}t^{-1/2}dt$ …(by substitution) $= \frac{1}{2h^3}\int_0^{\infty} e^{-t}t^{1/2}dt = \frac{1}{2h^3}\Gamma\frac{3}{2}$ …(by definition) $= \frac{1}{2h^3}\cdot\left(\frac{1}{2}\right)\sqrt{\pi}$ $\left[\because \Gamma\frac{3}{2} = \frac{1}{2}\Gamma\frac{1}{2} = \frac{1}{2}\sqrt{\pi}\right]$ $I = \frac{\sqrt{\pi}}{4h^3}$	
b.	Evaluate $\int_0^{\pi} x\sin^6 x\ dx$	

Soln. : Consider, $I = \int_{0} x \sin^{6} x \, dx$ ...(1)

$$I = \int_{0}^{\pi} (\pi - x) \sin 6 (\pi - x) \, dx$$

$$\ldots \left[\because \int_{0}^{a} f(x)\, dx = \int_{0}^{a} f(a-x)\, dx \right]$$

$$I = \int_{0}^{\pi} (\pi - x) \sin 6 x \, dx \; [\because \sin(\pi - x) = \sin x]$$

$$I = \int_{0}^{\pi} (\pi \sin 6 x - x \sin 6 x)\, dx$$

$$= \pi \int_{0}^{\pi} \sin 6 x \, dx - \int_{0}^{\pi} x \sin 6 x \, dx$$

$$I = \pi \int_{0}^{\pi} \sin 6 x \, dx - I$$

...[From Equation (1)]

$$\therefore 2I = \pi \int_{0}^{\pi} \sin^{6} x \, dx = \pi \cdot 2 \int_{0}^{\pi/2} \sin^{6} x \, dx$$

$$\left[\int_{0}^{\pi} \sin^{m} x \, dx = 2 \int_{0}^{\pi/2} \sin^{m} x \, dx \text{ when } m = \text{even} \right]$$

$$\therefore \quad I = \pi \int_{0}^{\pi/2} \sin^{6} x \, dx = \pi \int_{0}^{\pi/2} \sin^{6} x \cos^{0} x \, dx$$

$$I = \pi \cdot \frac{1}{2} \beta \left(\frac{7}{2}, \frac{1}{2} \right) = \frac{\pi}{2} \frac{\Gamma\left(\frac{7}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(4)}$$

$$\left[\because \int_{0}^{\pi/2} \sin^{p} \theta \cos^{q} \theta \, d\theta = \frac{1}{2} \left(\frac{p+1}{2}, \frac{q+1}{2} \right) \right]$$

$$I = \frac{\pi}{2} \frac{\frac{5}{2} \cdot \frac{3}{2} \frac{1}{2} \sqrt{\pi} \sqrt{\pi}}{3\,!}$$

$$\therefore \quad I = \frac{5}{32} \pi^{2}$$

c. Show that : $\int_{0}^{\pi/2} \frac{\log\left(1 + a \sin^{2} x\right)}{\sin^{2} x} . dx = \pi \left[\sqrt{1 + a} - 1 \right]$

Step (I) :

$$\text{Let } I(a) = \int_0^{\pi/2} \frac{\log(1 + a\sin^2 x)}{\sin^2 x} \cdot dx \qquad \text{...(1)}$$

Applying D.U.I.S., w.r.t. a, we get,

$$I'(a) = \int_0^{\pi/2} \frac{\partial}{\partial a}\left[\frac{\log(1 + a\sin^2 x)}{\sin^2 x}\right] \cdot dx$$

$$= \int_0^{\pi/2} \frac{1}{\sin^2 x}\left[\frac{\partial}{\partial a}\log(1 + a\sin^2 x)\right] \cdot dx$$

$$\therefore I'(a) = \int_0^{\pi/2} \frac{1}{\sin^2 x} \frac{1}{(1 + a\sin^2 x)} \cdot \sin^2 x \, dx$$

$$= \int_0^{\pi/2} \frac{1}{1 + a\sin^2 x} dx = \int_0^{\pi/2} \frac{1}{1 + \frac{a}{\text{cosec}^2 x}} \cdot dx$$

$$= \int_0^{\pi/2} \frac{\text{cosec}^2 x \cdot dx}{\text{cosec}^2 x + a} = \int_0^{\pi/2} \frac{\text{cosec}^2 x \cdot dx}{\cot^2 x + (1 + a)}$$

(Replacing $\text{cosec}^2 x$ by $(1 + \cot^2 x)$ in denominator)

Let $\cot x = u \quad \therefore -\text{cosec}^2 x \cdot dx = du$

x	0	$\pi/2$
u	∞	0

$$\therefore I'(a) = \int_\infty^0 \frac{-du}{u^2 + (a+1)} = \int_\infty^0 \frac{du}{u^2 + (a+1)}$$

$$= \left[\frac{1}{\sqrt{a+1}} \tan^{-1} \frac{u}{\sqrt{a+1}}\right]_0^\infty$$

$$\therefore I'(a) = \frac{\pi}{2\sqrt{a+1}} - 0$$

Step (II) : Integrating with respect to a, we get

$$I(a) = \frac{\pi}{2} \int \frac{1}{\sqrt{a+1}} da + A$$

$$I(a) = \frac{\pi}{2}[2\sqrt{a+1}] + A$$

$$\therefore I(a) = \pi\sqrt{a+1} + A \qquad \text{...(2)}$$

Step (III) : Putting $a = 0$ in Equation (1) to find A, we get,

$I(0) = 0$ and from Equation (2)

$I(0) = \pi + A \qquad \therefore \pi + A = 0$

$\therefore A = -\pi$

$\therefore$ From Equation (2),

$$I(a) = \pi\sqrt{a+1} - \pi = \pi[\sqrt{a+1} - 1]$$

d.	Show that : $\int_0^\infty \frac{\sin x}{x}.dx = \frac{\pi}{2}$	

Step (I) : Consider,

$$I(a) = \int_0^\infty e^{-ax} \cdot \frac{\sin x}{x} \cdot dx \quad ...(1)$$

Applying D.U.I.S., w.r.t. a, we get,

$$I'(a) = \int_0^\infty \frac{\partial}{\partial a}\left[e^{-ax} \cdot \frac{\sin x}{x}\right] \cdot dx$$

$$= \int_0^\infty \frac{\sin x}{x} \times \frac{\partial}{\partial a}(e^{-ax}) \cdot dx$$

$$= \int_0^\infty \frac{\sin x}{x}(-x\,e^{-ax}) \cdot dx$$

$$\therefore\ I'(a) = -\int_0^\infty e^{-ax} \sin x \cdot dx$$

[evaluating the integral by using the standard Formula of $\int e^{ax} \sin bx \cdot dx$]

$$= -\left[\frac{1}{a^2+1}(-a\,e^{-ax} \sin x - e^{-ax} \cos x)\right]_0^\infty$$

$$\therefore\ I'(a) = -\frac{1}{a^2+1} \qquad (\because\ e^{-\infty} = 0)$$

Step (II) : Integrating with respect to a,

$$I(a) = -\int \frac{1}{a^2+1}\,da + A$$

$$I(a) = -\tan^{-1} a + A \quad ...(2)$$

Step (III) : To find A, we put $a = \infty$ in Equation (1)

We get, $I(\infty) = 0$

and from Equation (2),

$$I(\infty) = -\frac{\pi}{2} + A$$

$$\therefore\ 0 = -\frac{\pi}{2} + A$$

$$\therefore\ A = \frac{\pi}{2}$$

$\therefore$ From Equation (2), $I(a) = -\tan^{-1} a + \frac{\pi}{2}$

i.e. $\int_0^\infty e^{-ax} \cdot \frac{\sin x}{x} \cdot dx = \frac{\pi}{2} - \tan^{-1} a$

We put $a = 0$, on both sides and we get,

$$\int_0^\infty \frac{\sin x}{x} \cdot dx = \frac{\pi}{2}$$

e.	Find : $\frac{d}{dx}\left[erf(x) + erf_c(ax)\right]$	

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \cdot du$$

Applying Leibnitz' rule of D.U.I.S. w.r.t. x,

$$\frac{d}{dx}\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\left[\int_0^x \frac{\partial}{\partial x} e^{-u^2} \cdot du + e^{-x^2}\, b \frac{d}{dx}(x) - 0\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[0 + e^{-x^2} \cdot 1\right]$$

$$= \frac{2}{\sqrt{\pi}} e^{-x^2} \qquad \text{...(1)}$$

Again, $$\operatorname{erf}_c(ax) = \frac{2}{\sqrt{\pi}} \int_{ax}^{\infty} e^{-u^2} \cdot du$$

Applying D.U.I.S. w.r.t. x; we get,

$$\frac{d}{dx}\operatorname{erf}_c(ax) = \frac{2}{\sqrt{\pi}} \frac{d}{dx}\left[\int_{ax}^{\infty} e^{-u^2} \cdot du\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\int_{ax}^{\infty} \frac{\partial}{\partial x} e^{-u^2} \cdot du + 0 - e^{-a^2x^2} \frac{d}{dx}(ax)\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[0 + ae^{-a^2x^2}\right]$$

$$= \frac{-2a}{\sqrt{\pi}} e^{-a^2x^2} \qquad \text{...(2)}$$

From Equations (1) and (2),

$$\frac{d}{dx}\left[\operatorname{erf}(x) + \operatorname{erf}_c(ax)\right] = \frac{2}{\sqrt{\pi}} e^{-x^2} - \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$$

$$= \frac{2}{\sqrt{\pi}}\left[e^{-x^2} - ae^{a^2x^2}\right]$$

f. If $\phi(\alpha) = \int_{f(\alpha)}^{g(\alpha)} F(x, \alpha)\, dx,$ write the rule to find $\frac{d\phi}{d\alpha}$ and hence prove that,

$$\frac{d}{dx}\left[erf\,\sqrt{x}\right] = \frac{e^{-x}}{\sqrt{\pi x}}$$

$$\phi(\alpha) = \int_{f(\alpha)}^{g(\alpha)} F(x, \alpha) \cdot dx$$

By Leibnitz rule,

$$\phi'(\alpha) = \int_{f(\alpha)}^{g(\alpha)} \left[\frac{\partial}{\partial \alpha} F(x, \alpha)\right] \cdot dx + F[g(\alpha), \alpha] \cdot \frac{dg}{d\alpha} - F[f(\alpha), \alpha] \frac{df}{d\alpha}$$

$$\text{Now, erf}\left(\sqrt{x}\right) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\cdot dt$$

...(by definition)

$$\frac{d}{dx}\left[\text{erf}\left(\sqrt{x}\right)\right] = \frac{2}{\sqrt{\pi}}\frac{d}{dx}\left[\int_0^{\sqrt{x}} e^{-t^2}\cdot dt\right]$$

By the Leibnitz' rule

$$= \frac{2}{\sqrt{\pi}}\left[\int_0^{\sqrt{x}}\left[\frac{\partial}{\partial x}e^{-t^2}\right]dt + e^{-(\sqrt{x})2}\cdot\frac{d}{dx}\left(\sqrt{x}\right) - e^{-0}\cdot\frac{d}{dx}(0)\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[0 + e^{-x}\cdot\frac{1}{2\sqrt{x}} - 0\right]$$

$$\left[\frac{\partial}{\partial x}\left(e^{-t^2}\right) = 0 \because \text{x and t are independent}\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{1}{2\sqrt{x}}e^{-x}\right] = \frac{1}{\sqrt{\pi x}}e^{-x}$$