

## Numerical Differentiation:

Numerical differentiation and integration for set of tabulated values depends basically on the method of approximating a given or unknown function  $f(x)$  over a short interval by a polynomial  $P_n(x)$  which is then differentiated or integrated. This method can advantageously be employed in cases where a given function  $f(x)$  may be difficult or practically impossible to integrate or differentiate.

### (A) From Newton Gregory Forward Difference interpolation:

The interpolating polynomial using forward difference is,

$$y = P_n(x) = \sum_{r=0}^n \frac{u^{[r]}}{r!} \Delta^r y_0 + \frac{h^{n+1}}{(n+1)!} u^{[n+1]} f^{n+1}(\varepsilon) \quad \dots (1)$$

$$x_0 < \varepsilon < x_n \quad \dots (2)$$

$$\text{Where } u = \frac{x-x_0}{h}$$

$$\begin{aligned} \text{i.e. } y = P_n(x) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \dots \\ &\dots + \frac{h^{n+1}}{(n+1)!} u^{[n+1]} f^{n+1}(\varepsilon) \quad \dots (3) \end{aligned}$$

Differentiating (3) successively w.r.t.  $x$ , we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{h} \left[ \Delta y_0 + \frac{(2u-1)}{2!} \Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!} \Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!} \Delta^4 y_0 + \dots \right] \\ &\quad + \frac{h^n}{(n+1)!} f^{n+1}(\varepsilon) \frac{d}{du} [u^{[n+1]}] \quad \dots (4) \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{1}{h^2} \left[ \Delta^2 y_0 + \frac{(6u-6)}{3!} \Delta^3 y_0 + \frac{12u^2-36u+22}{4!} \Delta^4 y_0 + \dots \right] \\ &\quad + \frac{h^{n-1}}{(n+1)!} f^{n+1}(\varepsilon) \frac{d^2}{du^2} [u^{[n+1]}] \quad \dots (5) \end{aligned}$$

$$\begin{aligned} \frac{d^3y}{dx^3} &= \frac{1}{h^3} \left[ \Delta^3 y_0 + \frac{24u-36}{4!} \Delta^4 y_0 + \dots \right] \\ &\quad + \frac{h^{n-2}}{(n+1)!} f^{n+1}(\varepsilon) \frac{d^3}{du^3} [u^{[n+1]}] \quad \dots (6) \end{aligned}$$

In order to evaluate the derivatives at any point of the data, we substitute the corresponding value of  $u$  in the expressions (vi), (v) and (vi), where the remainder in each expression gives the order of error.

**Example:-** Assuming a polynomial of second degree approximates  $f(x)$  at  $(x_0, y_0), (x_1, y_1)$  and  $(x_2, y_2)$ , evaluate first derivatives at  $x_0, x_1, x_2$  and the second derivative at  $x_1$  with respect error terms.

$$Y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{h^3 u(u-1)(u-2)}{3!} f'''(\epsilon), \quad x_0 < \epsilon < x_2$$

Is the polynomial of second degree, where  $u = \frac{x-x_0}{h}$

$$\text{Therefore, } \frac{dy}{dx} = \frac{1}{h} [\Delta y_0 + \frac{2u-1}{2!} \Delta^2 y_0] + \frac{h^2}{3!} (3u^2 - 6u + 2) f'''(\epsilon). \quad (i)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} [\Delta^2 y_0] + \frac{h}{3!} (6-6) f'''(\epsilon) \dots \dots \dots (ii)$$

For  $x=x_0, u=0$ , thus from (i), we have

$$\begin{aligned} \left(\frac{dy}{dx}\right)_{x_0} &= \frac{1}{h} [\Delta y_0 - \frac{1}{2!} \Delta^2 y_0] + \frac{1}{3} h^2 f'''(\epsilon) \\ &= \frac{1}{2h} [2(y_1 - y_0) - (y_2 - 2y_1 + y_0)] + \frac{h^2}{3} f'''(\epsilon) \\ &= \frac{1}{2h} [-3y_0 + 4y_1 - y_2] + \frac{h^2}{3} f'''(\epsilon) \end{aligned}$$

$$\text{Thus error term} = \frac{h^2}{3} f'''(\epsilon)$$

For  $x=x_1, u=1$ , thus from (i), we get

$$\begin{aligned} \left(\frac{dy}{dx}\right)_{x_1} &= \frac{1}{h} [\Delta y_0 + \frac{1}{2!} \Delta^2 y_0] + \frac{1}{3} h^2 f'''(\epsilon) \\ &= \frac{1}{2h} (y_2 - y_0) - \frac{1}{3!} h^2 f'''(\epsilon) \end{aligned}$$

For  $x=x_2, u=2$

$$\text{Therefore, } \left(\frac{dy}{dx}\right)_{x_2} = \frac{1}{h} [\Delta y_0 + \frac{3}{2!} \Delta^2 y_0] + \frac{1}{3} h^2 f'''(\epsilon)$$

$$= \frac{1}{2h} [y_0 - 4y_1 + 3y_2] + \frac{1}{3} h^2 f''' (\epsilon)$$

To evaluate  $\frac{d^2y}{dx^2}$ ,  $x=x_1, u=1$ , thus from (ii), we get

$$\left(\frac{d^2y}{dx^2}\right)_{x_1} = \frac{1}{h^2} [\Delta^2 y_0] + 0$$

$$= \frac{1}{h^2} [y_2 - 2y_1 + y_0]$$

Similarly

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2} [y_2 - 2y_1 + y_0] + h f''' (\epsilon)$$

### (B) From Newton Gregory Backward Difference Formula :

The Interpolating formula as given by

$$y_{(-r)} = y_0 - r \nabla y_0 + \frac{r(r-1)}{2!} \nabla^2 y_0 - \frac{r(r-1)(r-2)}{3!} \nabla^3 y_0 + \dots$$

To find the derivatives, similar method as explained in (A) above, is employed.

**Example:** if a polynomial of second degree passes through  $(x_{-1}, x_{-2})$ ,  $(x_0, y_0)$ ,  $(x_1, y_1)$ , evaluate first derivative at  $x_{-1}$ , in terms of the given value of  $y_1$  using backward difference interpolation.

The interpolation polynomial of 2<sup>nd</sup> degree is,

$$Y = y_1 - r \Delta y_1 + \frac{r(r-1)}{2!} \nabla^2 y_1, \quad x = x_1 - rh$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dr} \cdot \frac{dr}{dx} = \pi r^2$$

$$\frac{-1}{h} \left[ -\nabla y_1 + \frac{2r-1}{2} \nabla^2 y_1 \right]$$

For  $x_{-1}, r=2$

$$\begin{aligned} \therefore \left(\frac{dy}{dx}\right)_{x_{-1}} &= -\frac{1}{h} \left[ -(y_0 - y_{-1}) + \frac{3}{2} (y_1 - 2y_0 - y_{-1}) \right] \\ &= \frac{1}{2h} (y_{-1} - 4y_0 + 3y_1) \end{aligned}$$

**(C) From striling's formula:**

The striling's interpolation in terms of central difference is [from result (37)].

$$Y=y_0+\mu\delta.y_0 + \frac{1}{2!} r^2\delta^2y_0 + \frac{1}{3!} r(r^2 - 1)\mu\delta^3y_0 + \frac{r^2(r^3-1)}{4!} \delta^4y_0 + \dots \text{ where } x = x_0 + rh$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dr} \cdot \frac{dr}{dx}$$

$$= \frac{1}{h} [\mu\delta y_0 + r\delta^2 y_0 + \frac{1}{3!} (3r^2 - 1)\mu\delta^3 y_0 + \frac{1}{4!} (4r^3 - 2r)\delta^4 y_0]$$

Thus, to evaluate at  $x= x_0$  ,  $r=0$

$$\therefore \left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h} [\mu\delta y_0 - \frac{1}{3!}\mu\delta y_0 \dots]$$

In similar way derivative at other points in the table can be evaluated. For higher derivatives, the method is the same as employed in(A) and (B).

**(D) For unevenly spaced points, we use Lagrange form**

$$F(x)=y=\sum y_i L_i + R_n$$
$$\prod (x-x_r)$$

Where,  $L_i = \frac{r \neq i}{n}$

$$\prod (x_i - x_r)$$

$$r=0$$

$$(r \neq i)$$

And  $R_n = \frac{f^{n+1}(\epsilon)}{(n+1)!} \pi (x - x_r)$

And follow the procedure illustrated in (A).

**Example 1:** evaluate  $(\frac{dy}{dx})_{0.1}$ ,  $(\frac{d^2y}{dx^2})_{0.1}$ , with errors of order  $h^3$

From the table:

|   |        |        |        |        |        |
|---|--------|--------|--------|--------|--------|
| x | 0.0    | 0.1    | 0.2    | 0.3    | 0.4    |
| y | 1.0000 | 1.9975 | 0.9900 | 0.9776 | 0.9604 |

Since error of order is required, we consider the polynomial of second degree for  $\frac{dy}{dx}$  and of third degree for  $\frac{d^2y}{dx^2}$

Difference table is :

| x   | y      | $\Delta$ | $\Delta^2$ | $\Delta^3$ |
|-----|--------|----------|------------|------------|
| 0.0 | 1.0000 |          |            |            |
| 0.1 | 0.9975 | -0.0025  |            |            |
| 0.2 | 0.9900 | -0.0075  | -0.0050    |            |
| 0.3 | 0.9776 | -0.0124  | -0.0049    | 0.0001     |
| 0.4 | 0.9604 | -0.0172  | -0.0048    | 0.0001     |

(1) For  $\frac{dy}{dx}$ , the interpolating polynomial of 2<sup>nd</sup> degree is  $[x_0=0.1] y = 0.9975 + u (-0.007) + \frac{u(u-1)}{2!}$

$(-0.0049) + \frac{h^3}{3!} u^{[3]} f'''(\epsilon)$  where  $u = \frac{x-x_0}{0.1}$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du} = \frac{1}{0.1} \left[ -0.0075 + \frac{2u-1}{2!} (-0.0049) \right] + \frac{h^2}{3!} + f'''(\epsilon)(3u^2 - 6u + 2)$$

For  $x=0.1$ .  $u=0$

$$\therefore (\frac{dy}{dx})_{0.1} = 10[-0.0075 - \frac{1}{2}(-0.0049)] + \frac{h^2}{3!} f'''(\epsilon) \cdot 2$$

$$= -0.0500 + \frac{h^2}{3} f'''(\epsilon)$$

$$= -0.0500 + O(h^2)$$

(2) For  $\frac{d^2y}{dx^2}$ , the interpolating polynomial 3<sup>rd</sup> degree is ( $x_0=0.1$ )

$$y=0.9975+u(-0.0075)+\frac{u(u-1)}{2!}(-0.0049)+\frac{u(u-1)(u-2)}{3!}(0.0001)+\frac{h^4}{4!}f^{iv}(\epsilon) u (u-1) (u-2) (u-3)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du} = \frac{1}{0.1} \left[ -0.0075 + \frac{2u-1}{2!}(-0.0049) + \frac{3u^2 - \epsilon u + 2}{3!}(0.0001) \right]$$

$$+ \frac{h^3}{4!}f^{iv}(\epsilon)[4u^3 - 18u^3 + 22u - 6]$$

$$\frac{d^2y}{dx^2} = \frac{1}{(0.1)^2}[-0.0049 + \frac{(6u-6)}{3!}(0.0001)] + \frac{h^2}{4!}f^{iv}(\epsilon)(12u^2 - 36u + 22)$$

For  $x=0.1$ ,  $u=0$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{0.1} = 100[-0.0049 - 0.0001] + \frac{h^2}{4!}f^{iv}(\epsilon)$$

$$= -0.5000 + \frac{11}{12}h^2f^{iv}(\epsilon)$$

$$= -0.5000 + O(h^2)$$

Example-2 for the data in the example 1, evaluate  $\left(\frac{dy}{dx}\right)_{0.2}$  by using central differences

By Stirling's formula,

$$Y = y_0 + ru\delta y_0 + \frac{1}{2!}r^2\delta^2 y_0 + \frac{1}{3!}r(r^2 - 1)\delta^3 y_0 + \dots$$

Using the table of previous example, we have for  $x_0 = 0.2$  and  $h = 0.1$   $y_0 = 0.9900$

$$\mu\delta y_0 = \frac{-0.0075 - 0.0124}{2} = -0.0098$$

$$\delta^2 y_0 = -0.0049$$

$\therefore$  Interpolation Polynomial is  $[x = x_0 - rh]$

$$y = 0.990 + (-0.0098)r + \frac{1}{2!}(-0.0049)r^2$$

$$\therefore \frac{dy}{dx} = \frac{dr}{dx} \frac{dy}{dr} = -\frac{1}{0.1} [-0.0098 + (-0.0049)r]$$

For  $x = 0.2$ ,  $r = 0$

$$\therefore \left(\frac{dy}{dx}\right)_{0.2} = 10 [0.0098] = 0.098$$

Example 3:- For the set of points (0, 2), (2, -2), (3, -1), evaluate  $\left(\frac{dy}{dx}\right)_2$

Since the points are unevenly spaced, we have by lagrange's formula,

$$\begin{aligned}
 y &= (2) \frac{(x-2)(x-3)}{(-2)(-3)} + (-2) \frac{(x-0)(x-3)}{(2)(-1)} + (-1) \frac{(x-0)(x-2)}{(3)(1)} \\
 &= \frac{1}{8} (x-2)(x-3) + x(x-3) - \frac{1}{8} x(x-2) \\
 \frac{dy}{dx} &= \frac{1}{8} [2x-5] + (2x-3) - \frac{1}{8} x(x-2) \\
 &= \frac{1}{8} [6x-12] = 2x-4
 \end{aligned}$$

$$\left(\frac{dy}{dx}\right)_2 = 2(2) - 4 = 0$$

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1. Find  $\left(\frac{dy}{dx}\right)_{2.5}$  and  $\left(\frac{d^2y}{dx^2}\right)_{2.5}$  from the table:

|   |        |        |        |        |        |
|---|--------|--------|--------|--------|--------|
| X | 2.50   | 2.55   | 2.60   | 2.65   | 2.70   |
| Y | 1.5811 | 1.5969 | 1.6125 | 1.6279 | 1.6431 |

2. Determine the limits to errors when  $x=0.6$ , in representing  $f(x)=\sin x$  by polynomial of degree five in the range  $x=0(0.1) 0.5$ . use Gregory newton formula.

3. From the table of Bessel function  $J_n(1)$  given, evaluate  $J_{\frac{1}{3}}(1)$  and

$\left[\frac{d}{dn} (J_n(1))\right]_{n=\frac{1}{4}}$  using stirling's formula.

|          |         |               |               |               |        |               |               |               |        |
|----------|---------|---------------|---------------|---------------|--------|---------------|---------------|---------------|--------|
| X        | -1      | $\frac{3}{6}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | 0      | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{3}{6}$ | 1      |
| $J_n(1)$ | -0.4401 | 0.0447        | 0.4311        | 0.6694        | 0.7652 | 0.7522        | 0.6714        | 0.5587        | 0.4401 |

[Ans. 0.7309, 0.657]

4. Given  $\log 280=2.4472$ ,  $\log 281=2.4487$ ,  $\log 283=2.4518$   $\log 286 = 2.4564$ , find  $\left[\frac{d}{dx} (\log x)\right]_{x=280}$

[Ans. 0.00742]

5. Find  $\left(\frac{dy}{dx}\right)_{5.2}$ , by using stirling's formula, if

|   |         |         |         |         |         |         |         |
|---|---------|---------|---------|---------|---------|---------|---------|
| X | 4.9     | 5.0     | 5.1     | 5.2     | 5.3     | 5.4     | 5.5     |
| y | 134.290 | 148.413 | 164.022 | 181.272 | 200.337 | 221.406 | 244.692 |

[Ans. 018.148]

6. If  $y=\sqrt{x}$ , find  $\frac{dy}{dx}, \frac{d^2y}{dx^2}$  at  $x= 2.5$ , from the table:

|            |         |         |         |         |         |         |
|------------|---------|---------|---------|---------|---------|---------|
| x          | 2.50    | 2.55    | 2.60    | 2.65    | 2.70    | 2.75    |
| $\sqrt{x}$ | 1.58114 | 1.59687 | 1.61245 | 1.62788 | 1.64317 | 1.65831 |

[Ans. -0.3162, -0.0632]

7. the table for bassel function  $j_0(x)$  is given by

|          |       |        |        |        |        |
|----------|-------|--------|--------|--------|--------|
| X        | 0.0   | 0.1    | 0.2    | 0.3    | 0.4    |
| $j_0(x)$ | 1.000 | 0.9975 | 0.9900 | 0.9776 | 0.9004 |

Evaluate the following derivatives, with errors of the indicated order using backward, central and forward difference expansion: -

(a).  $\left[\frac{dj_0}{dx}\right]_{x=0.1}$  ;  $e=O(h^2)$

(b).  $\left[\frac{d^2j_0}{dx^2}\right]_{x=0.1}$  ;  $e=O(h^2)$

(c).  $\left[\frac{d^3j_0}{dx^3}\right]_{x=0}$  ;  $e=O(h)$

(d).  $\left[\frac{dj_0}{dx}\right]_{x=0.4}$  ;  $e=O(h)$

(e).  $\left[\frac{d^3j_0}{dx^3}\right]_{x=0.2}$  ;  $e=O(h^2)$

Ans. (a) -0.8500      (b) -0.5000      (c) 0.1000      (d) -0.1720      (e) -0.4900.j

8. from the following table, find the number of students who obtained less than 45 marks.

|                 |       |       |       |       |       |
|-----------------|-------|-------|-------|-------|-------|
| Marks           | 30-40 | 40-50 | 50-60 | 60-70 | 70-80 |
| No. of students | 31    | 35    | 51    | 42    | 31    |

[ans. 47.868=48]

9. the velocity (V ft./sec) of a vehicle is given at equal intervals of time (T). Calculate the acceleration when T=7 and T=10, from the table

|   |       |       |       |       |       |       |       |
|---|-------|-------|-------|-------|-------|-------|-------|
| T | 5     | 6     | 7     | 8     | 9     | 10    | 11    |
| V | 17.16 | 19.07 | 28.80 | 22.31 | 23.57 | 24.57 | 25.26 |

And also determine the distance covered from 5 sec, to 12 sec.

[ans. 1.63'/sec<sup>2</sup>, 0.85'/sec<sup>2</sup>, 131.65']

10. A rocket is launched from the ground. Its acceleration is registered during first 80 seconds and is given by the table:-

|       |       |       |       |       |       |       |      |       |       |
|-------|-------|-------|-------|-------|-------|-------|------|-------|-------|
| $t =$ | 0     | 10    | 20    | 30    | 40    | 50    | 60   | 70    | 80    |
| $a =$ | 30.00 | 31.63 | 33.34 | 35.47 | 37.75 | 40.33 | 43.2 | 46.69 | 50.67 |

Find the velocity and the height of the rocket at time t=80 seconds. [3087 meter, height = 172.75 km]

11. Calculate sinh 1.1 from the table; using differences

|           |        |        |        |        |        |        |
|-----------|--------|--------|--------|--------|--------|--------|
| $x$       | 1      | 1.1    | 1.2    | 1.3    | 1.4    | 1.5    |
| $\cos hx$ | 1.5431 | 1.6685 | 1.8107 | 1.9709 | 2.1504 | 2.3524 |